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Lecture 24

I. K. Rana 7/2/11

Measure and Integration

$$X \times Y \longrightarrow X$$

$$\mathcal{A} \otimes \mathcal{B} \qquad \mathcal{A}$$

$$p_X : X \times Y \longrightarrow X$$

$$p_X(x,y) = x \quad \forall \ (x,y) \in X \times Y$$

p_X is $\mathcal{A} \otimes \mathcal{B}$ measurable:

$\forall \ A \in \mathcal{A}$, $p_X^{-1}(A) \in \mathcal{A} \otimes \mathcal{B}$?

<u>Note</u>

$$p_X^{-1}(A) = \{(x,y) \in X \times Y \mid x \in A\}$$

$$= A \times Y \in \mathcal{R} \subseteq \mathcal{A} \otimes \mathcal{B}.$$

p_X is a $\mathcal{A} \otimes \mathcal{B}$ measurable

$$p_Y : X \times Y \longrightarrow Y$$

$$p_Y(x,y) = y \quad \forall\ (x,y) \in X \times Y$$

$\forall \quad B \in \mathcal{B}$,

$$p_Y^{-1}(B) = \{(x,y) \mid y \in B\}$$

$$= X \times B \in \mathcal{R} \subseteq \mathcal{A} \otimes \mathcal{B}$$

$\Longrightarrow \quad p_Y$ is $\mathcal{A} \otimes \mathcal{B}$ measurable

Let $\mathcal{S}$ be a σ-algebra of subsets of $X \times Y$ such that both

$$p_X : X \times Y \longrightarrow X$$

$$p_Y : X \times Y \longrightarrow Y$$

are measurable.

Show $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{S}$?

Note $A \times B \in \mathcal{R}$, then

$$A \times B = (A \times Y) \cap (X \times B)$$

$$= p_X^{-1}(A) \cap p_Y^{-1}(B) \in \mathcal{S}$$

$\Rightarrow \quad \mathcal{R} \subseteq \Sigma.$

~~**Claim**: Σ is a σ-algebra.~~

~~(i) $\phi = p_X^{-1}(\phi) \in$~~

$\Rightarrow \quad \mathcal{A} \otimes \mathcal{B} \subseteq \Sigma.$

$$X \qquad Y \qquad X \times Y$$

$$\mathcal{C} \qquad \mathcal{D} \qquad \mathcal{C} \times \mathcal{D}$$

$$\mathcal{S}(\mathcal{C}) \qquad \mathcal{S}(\mathcal{D}) \qquad \mathcal{S}(\mathcal{C} \times \mathcal{D})$$

Is $\mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D}) = \mathcal{S}(\mathcal{C} \times \mathcal{D})$?

<u>Note</u>

$$\mathcal{C} \subseteq \mathcal{S}(\mathcal{C})$$

$$\mathcal{D} \subseteq \mathcal{S}(\mathcal{D})$$

$$\underline{\mathcal{C} \times \mathcal{D}} \subseteq \mathcal{S}(\mathcal{C}) \times \mathcal{S}(\mathcal{D})$$

$$\subseteq \underline{\mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D})}$$

$$\Rightarrow \underline{\mathcal{S}(\mathcal{C} \times \mathcal{D}) \subseteq \mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D})}$$

Given $X = \bigcup_{i=1}^{\infty} C_i, \quad C_i \in \mathcal{C}$

$Y = \bigcup_{j=1}^{\infty} D_j, \quad D_j \in \mathcal{D}$

$\Longrightarrow \quad \mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D}) = \mathcal{S}(\mathcal{C} \times \mathcal{D})$

Only show

$$\mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D}) \subseteq \mathcal{S}(\mathcal{C} \times \mathcal{D})$$

$\left[\begin{array}{l} p_X \text{ is } \mathcal{S}(\mathcal{C} \times \mathcal{D}) \text{ measurable} \\ p_Y \text{ is } \mathcal{S}(\mathcal{C} \times \mathcal{D}) \text{ measurable} \end{array}\right.$

$$p_X : X \times Y \longrightarrow X$$

$$\mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D}) \qquad \mathcal{S}(\mathcal{C})$$

p_X is $\mathcal{S}(\mathcal{C} \times \mathcal{D})$ - msb ?

~~tst~~ $\forall\ A \in \mathcal{S}(\mathcal{C}) \Rightarrow p_X^{-1}(A) \in \mathcal{S}(\mathcal{C} \times \mathcal{D})$

$$p_X^{-1}(A) = A \times Y = A \times \left(\bigcup_{j=1}^{\infty} D_j\right),\ D_j \in \mathcal{D}$$

$$= \bigcup_{j=1}^{\infty} (\underline{A \times D_j}) \qquad \left[A \times D_j \in \mathcal{C} \times \mathcal{D} \text{ if } A \in \mathcal{C} \right]$$

If $A \in \mathcal{C}$, $p_X^{-1}(A) \in \mathcal{S}(\mathcal{C} \times \mathcal{D})$

$$\mathcal{U} = \{ A \in \mathcal{S}(\mathcal{C}) \mid p_X^{-1}(A) \in \mathcal{S}(\mathcal{C} \times \mathcal{D}) \}$$

We know

$$\mathcal{C} \subseteq \mathcal{U} \subseteq \mathcal{S}(\mathcal{C})$$

To show $\mathcal{U} = \mathcal{S}(\mathcal{C})$, enough to show $\mathcal{U}$ is a σ-algebra,

(i) $\phi \in \mathcal{U}$, $\phi = p_X^{-1}(\phi)$

$X \times Y = p_X^{-1}(X)$,

(ii) $A \in \mathcal{U} \Rightarrow p_X^{-1}(A) \in \mathcal{S}(\mathcal{C} \times \mathcal{D})$

$\Rightarrow (p_X^{-1}(A^c))^c \in \mathcal{S}(\mathcal{C} \times \mathcal{D})$

$$\Rightarrow \left(p_x^{-1}(A^c)\right)^c \in \Sigma(\mathcal{C}\times\mathcal{D})$$

$$\Rightarrow A^c \in \mathcal{U}$$

(iii) $\underset{i\geq 1}{A_i} \in \mathcal{U} \Rightarrow p_x^{-1}(A_i) \in \Sigma(\mathcal{C}\times\mathcal{D})$

$$\Rightarrow \bigcup_{i=1}^{\infty} p_x^{-1}(A_i) \in \Sigma(\mathcal{C}\times\mathcal{D})$$

$$\parallel$$

$$p_x^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{U}.$$

$\mathbb{R}$

$\mathcal{U}$ = The collection of open sets

$\mathcal{B}_{\mathbb{R}} = \mathcal{S}(\mathcal{U})$

$\mathbb{R}^2,\ \mathcal{U}_{\mathbb{R}^2},\ \mathcal{B}_{\mathbb{R}^2} = \mathcal{S}(\mathcal{U}_{\mathbb{R}^2})$

<u>Claim</u> $\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$?

$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{S}(\mathcal{U}) \otimes \mathcal{S}(\mathcal{U})$

$X = \mathbb{R} = Y \quad \Big| \quad \mathcal{S}(\mathcal{U} \times \mathcal{U})$

$\mathcal{C} = \mathcal{D} = \mathcal{U} \quad \Big| \quad = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$

$\mathcal{U} \times \mathcal{U}$ $\qquad$ $U_1 \times U_2 \in \mathcal{U} \times \mathcal{U}$

U_1 and U_2 open

$\underline{E \in \mathcal{U}_{\mathbb{R}^2}} \Rightarrow E = \bigcup_{i=1}^{\infty} E_i,$

$E_i \in \mathcal{U} \times \mathcal{U}$

[$\mathbb{R}^2$ is secon countable

& $\{U_1 \times U_2\}$ form a base for open

sets in $\mathbb{R}^2$]

$\Sigma(\mathcal{U} \times \mathcal{U}) = \Sigma(\mathcal{U}_{\mathbb{R}^2})$

$\underline{= \mathcal{B}_{\mathbb{R}^2}}$

X $\qquad\qquad$ Y

$\mathcal{A}$ $\qquad\qquad$ $\mathcal{B}$

$$\mathcal{R} = \{ A \times B \mid A \in \mathcal{A} \mid B \in \mathcal{B} \}$$

rectangles in $X \times Y$

$\mathcal{R}$ is not a σ-algebra.

$$\Sigma(\mathcal{R}) := \underline{\mathcal{A} \otimes \mathcal{B}}.$$

$$X \qquad\qquad Y$$

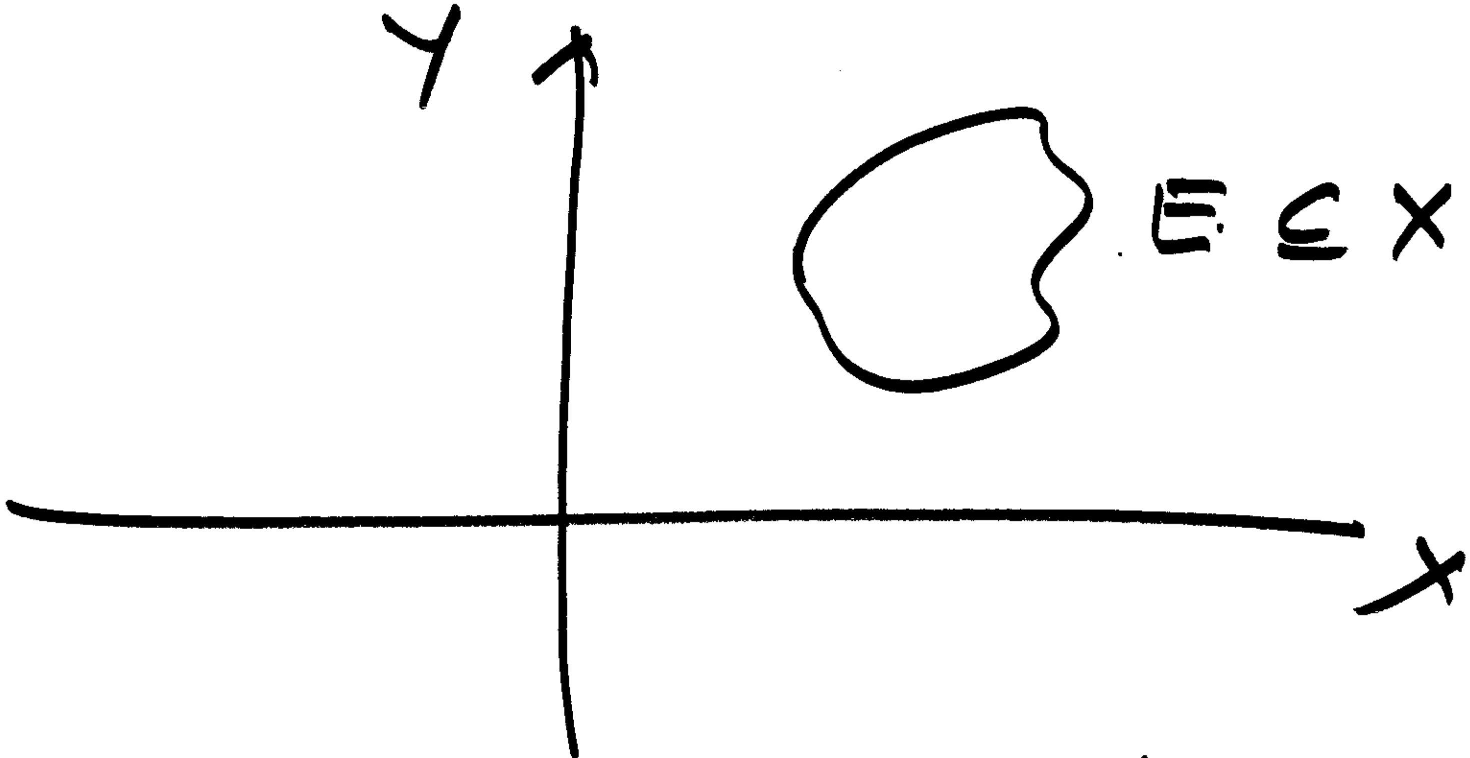
$$\underline{\mathcal{C}} \qquad\qquad \underline{\mathcal{D}}$$

$$\mathcal{C} \times \mathcal{D} \subseteq \mathcal{P}(X \times Y)$$

$$\mathcal{S}(\mathcal{C} \times \mathcal{D}) = \mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D})$$

if
$X = \bigcup C_i$, $C_i \in \mathcal{C}$

$Y = \bigcup D_j$, $D_j \in \mathcal{D}$.



$E \subseteq X \times Y$

$\mu(A), \quad A \in \mathcal{A}$

$\nu(B), \quad B \in \mathcal{B}$